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Elliptic operators, topology and asymptotic methods

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12 Witten's approach to Morse theory

This chapter and the next treat some additional topics, and we will allow ourselves a few more facts from functional analysis. In particular we will need some information on unbounded symmetric operators: the theory of the deficiency indices and self-adjoint extensions, Friedrichs' extension theorem and the spectral theorem. This material can all be found in Dunford and Schwartz [27].

In Chapter 4 we defined a *Dirac complex* over a compact Riemannian manifold M, and we proved the Hodge theorem, that the cohomology of such a complex is represented by harmonic sections. According to (9.6), the index of the associated Dirac operator is just the Euler characteristic of the complex, that is the alternating sum of the dimensions of the various cohomology groups. If we define the *Betti numbers* of the Dirac complex (S,d) by

$$\beta_j = \dim H^j(S,d), \tag{12.1}$$

then

$$\operatorname{index}(d+d^*) = \Sigma(-1)^j \beta_j.$$

The *Morse inequalities* are a system of inequalities that allow one to estimate the individual Betti numbers β_j. In analysis, the Morse inequalities arise as follows. Suppose that ϕ is a smooth rapidly decreasing positive function on $\mathbb{R}^+$ with $\phi(0) = 1$. Then by (5.8) and (6.12), the operator $\phi(D^2)$ (where D is the Dirac operator) is of trace class. Set

$$\mu_j = \operatorname{Tr}(\phi(D^2)|_{S_j}). \tag{12.2}$$

Then:

(12.3) <u>Proposition</u>: With the hypotheses above, the numbers (μ_j) and (β_j) satisfy the following system of inequalities (known as the *Morse inequalities*):

$$\mu_0 \geq \beta_0$$

$$\mu_1 - \mu_0 \geq \beta_1 - \beta_0$$

$$\mu_2 - \mu_1 + \mu_0 \geq \beta_2 - \beta_1 + \beta_0$$

etc., and finally an equality

$$\Sigma(-1)^j \mu_j = \Sigma(-1)^j \beta_j.$$

Proof: By the Hodge theorem (4.2), β_j is equal to the dimension of the kernel of D^2 on sections of S_j. Since the spectrum of D^2 is discrete, there is a smooth function $\tilde{\phi}$ on $\mathbb{R}^+$ which is positive, rapidly decreasing, with $\tilde{\phi}(0) = 1$ and $\tilde{\phi}(\lambda) = 0$ for all non-zero eigenvalues λ of D^2; there is no loss of generality in assuming also that $\tilde{\phi} \leq \phi$. Then $\beta_j = \mathrm{Tr}(\tilde{\phi}(D^2)|_{S_j})$, so that $\mu_j - \beta_j = \mathrm{Tr}((\phi-\tilde{\phi})(D^2)|_{S_j})$.

We may write the function $\phi - \tilde{\phi}$ in the form

$$(\phi-\tilde{\phi})(\lambda) = \lambda(\psi(\lambda))^2$$

where ψ is positive and rapidly decreasing, since $\phi - \tilde{\phi}$ is rapidly decreasing, vanishes at zero and is differentiable there. So we may write $(\phi-\tilde{\phi})(D^2) = D^2(\psi(D^2))^2$. Now we make a trace argument exactly as in the proof of (8.6). We have $D^2 = dd^* + d^*d$, and

$$\mathrm{Tr}(dd^*(\psi(D^2))^2|_{S_j})$$

$$= \mathrm{Tr}(\psi(D^2)|_{S_j} dd^* \psi(D^2)|_{S_j})$$

$$= \mathrm{Tr}(d^*(\psi(D^2))^2|_{S_j} d)$$

$$= \mathrm{Tr}(d^*d(\psi(D^2))^2|_{S_{j-1}}).$$

Therefore

$$(\mu_j-\beta_j) - (\mu_{j-1}-\beta_{j-1}) + (\mu_{j-2}-\beta_{j-2}) - \dots$$

$$= \mathrm{Tr}(d^*d(\psi(D^2))^2{}_{|S_j}).$$

If j equals the top dimension of the complex, then this is zero. In general, write

$$d^*d(\psi(D^2))^2{}_{|S_j} = A^*A, \text{ where } A = d\psi(D^2)_{|S_j}.$$

Now A is a trace-class operator, so we may write in any orthonormal basis (e_i) for $L^2(S)$

$$\mathrm{Tr}(A^*A) = \sum_i \langle A^*Ae_i, e_i\rangle$$

$$= \sum \|Ae_i\|^2 \geq 0.$$

Therefore $(\mu_j-\beta_j) - (\mu_{i-1}-\beta_{j-1}) + (\mu_{j-2}-\beta_{j-2}) - \dots \geq 0$, and the result follows. □

From now on, we will consider only the case of the de Rham complex.

(12.4) <u>Definition</u>: A smooth function $h : M \to \mathbb{R}$ is called a *Morse function* on M if at its critical points (that is, the points where the first derivatives Dh vanish) the matrix D^2h of second derivatives is non-singular.

Clearly the critical points of a Morse function are isolated, so there are only finitely many of them. Each critical point has an *index*, defined to be the number of negative eigenvalues of the (symmetric) matrix of second derivatives. The idea of Morse theory is that the critical points of a Morse function on M should somehow model the cohomology of M. Witten's approach to Morse theory is based on perturbing the de Rham complex in such a way that the operators involved concentrate near the critical points of h. Namely, the *perturbed exterior derivative* d_s (depending on the parameter $s \in \mathbb{R}^+$) is defined by

$$d_s\omega = e^{-sh}d(e^{sh}\omega) = d\omega + s\,dh \wedge \omega \,. \tag{12.5}$$

Its adjoint is given by

$$d_s^*\omega = e^{sh}d^*(e^{-sh}\omega) = d^*\omega - s\,dh \lrcorner\, \omega \,; \tag{12.6}$$

this agrees with the calculation of (2.12), that interior multiplication is minus the adjoint of exterior multiplication. The analogue of the Dirac operator is

$$D_s = d_s + d_s^* = D + sH \tag{12.7}$$

where H is the endomorphism of the exterior bundle given by $(dh\wedge) - (dh \lrcorner)$.

(12.8) Lemma: With the above notations

i) H^2 is the endomorphism given by multiplication by $|dh|^2$;

ii) $HD + DH$ is a zero'th-order operator, i.e. an endomorphism of the exterior bundle.

Proof: Choose a local orthonormal framing (e_α) of the tangent bundle to M. We will use the following fact, which may be verified by direct computation: the operator

$$e_\alpha \lrcorner\; e_\beta\wedge \quad + e_\beta \wedge e_\alpha \lrcorner$$

is just given by multiplication by $-(e_\alpha, e_\beta) = -\delta_{\alpha\beta}$. Now $H^2 = -(dh \wedge dh \lrcorner + dh \lrcorner\, dh\wedge)$, so this equals $|dh|^2$. As for $HD + DH$, if we write $dh = h_\alpha e_\alpha$, then

$$\begin{aligned}(HD+DH)\omega &= h_\alpha(e_\alpha\wedge - e_\alpha\lrcorner)(e_\beta\wedge + e_\beta\lrcorner)\nabla_\beta\omega + (e_\beta\wedge + e_\beta\lrcorner)\nabla_\beta(h_\alpha(e_\alpha\wedge - e_\alpha\lrcorner)\omega)\\ &= h_\alpha[(e_\alpha\wedge - e_\alpha\lrcorner)(e_\beta\wedge + e_\beta\lrcorner) + (e_\beta\wedge + e_\beta\lrcorner)(e_\alpha\wedge - e_\alpha\lrcorner)]\nabla_\beta\omega\\ &\quad + \text{endomorphism.}\end{aligned}$$

Now

$$(e_\alpha\wedge - e_\alpha\lrcorner)(e_\beta\wedge + e_\beta\lrcorner) + (e_\beta\wedge + e_\beta\lrcorner)(e_\alpha\wedge - e_\alpha\lrcorner)$$

$$= e_\alpha\wedge e_\beta\lrcorner - e_\alpha\lrcorner\, e_\beta\wedge - e_\beta\wedge e_\alpha\lrcorner + e_\beta\lrcorner\, e_\alpha\wedge$$

$$= \delta_{\alpha\beta} - \delta_{\alpha\beta} = 0. \qquad \square$$

Now we will look at the asymptotics of Witten's perturbed de Rham complex as $s \to \infty$. Eventually, we will need to choose a special metric on M that is nicely related to the Morse function h; but we can do the first part of the calculation without making this special choice.

We will also need to know that the basic elliptic theory of Chapter 3, and the results (5.5) - (5.8), extend without change to the operator D_s. Indeed, the operator D_s differs from the standard Dirac operator D only by a zero order perturbation, and the diligent reader will already have checked in Exercise 4.2 that the results of Chapter 3 extend to this situation; the proofs of (5.5) - (5.8) go over verbatim. As an example, let us verify the Garding inequality for D_s. By Lemma (12.8),

$$D_s^2 = D^2 + s(HD+DH) + s^2H^2$$

$$= D^2 + L$$

where L is an operator of order zero. Therefore

$$\|D_s\omega\|^2 = \|D\omega\|^2 + \langle L\omega,\omega\rangle \geq \|D\omega\|^2 - C_1\|\omega\|^2$$

for some constant C_1. Therefore

$$(1+C_1)(\|D_s\omega\|^2 + \|\omega\|^2) \geq \|D\omega\|^2 + \|\omega\|^2 \geq \frac{1}{C_2}\|\omega\|_1^2$$

by the usual Garding inequality (3.14); the Garding inequality for D_s follows.

Notice that the norm of L is of order s^2, so that the constant appearing in the Garding inequality is bounded by a polynomial in s. The same is true of the constant appearing in the elliptic estimate.

(12.9) We begin our asymptotic calculation of Witten's complex by fixing a number $\rho > 0$ and choosing a positive even function $\phi \in S(\mathbb{R})$ with $\phi(0) = 1$ and such that the Fourier transform $\hat{\phi}$ is supported within the interval $[-\rho,\rho]$. According to (12.3), the Betti numbers of M satisfy the Morse inequalities relative to the numbers $\mu_j = \mathrm{Tr}(\phi(D_s)|_{\Lambda^j T^*M})$. (Note that since ϕ is even, $\phi(D_s)$ can in fact be written as a function of D_s^2, so (12.3) is applicable). We investigate the asymptotics of $\phi(D_s)$ as $s \to \infty$. First, we work on the complement of a neighbourhood of the set of critical points of h. Let us denote this set of critical points Crit(h).

(12.10) Lemma: On the complement of a 2ρ-neighbourhood of Crit(h), the smoothing kernel of $\phi(D_s)$ (which exists by the analogue of (5.8) for the operator D_s) tends uniformly to zero as $s \to \infty$.

Proof: Since M is a compact manifold, there is a constant C such that $|\nabla h(x)| \geq C > 0$ for all x in the complement of a ρ-neighbourhood of Crit(h). Now by the formula

$$D_s^2 = D^2 + s(HD + DH) + s^2H^2$$

and Lemma (12.8) we find that for s large

$$\langle D_s^2\omega,\omega\rangle \geq \tfrac{1}{2}C^2s^2\|\omega\|^2 \qquad (*)$$

provided that ω is supported in the complement of such a neighbourhood. Now let $\mathbb{N}$ denote the Hilbert space of L^2 differential forms on M that vanish on a ρ-neighbourhood of Crit(h). Formula (*) shows that D_s^2 is a positive formally self-adjoint operator on $\mathbb{N}$. It therefore has a self-adjoint extension A on $\mathbb{N}$ satisfying the same positivity condition, by Friedrichs' extension theorem ([27], sec. XII.5). Now we will show that if ω is supported in the complement of a 2ρ-neighbourhood of Crit(h), then

$$\phi(D_s)\omega = \phi(\sqrt{A})\omega.$$

To do this we use finite propagation speed (5.5) for the operator D_s. Consider the time-dependent differential form

$$\omega_t = \cos(tD_s)\omega = \tfrac{1}{2}(e^{itD_s} + e^{-itD_s})\omega.$$

Clearly, ω_t is a solution to the partial differential equation

$$\frac{\partial^2 \omega_t}{\partial t^2} + D_s^2 \omega_t = 0$$

with initial condition $\omega_0 = \omega$, $\dot{\omega}_0 = 0$; in fact it is the unique solution, as one can easily check by verifying that the 'energy'

$$\left\|\frac{\partial \omega_t}{\partial t}\right\|^2 + \langle D_s^2 \omega_t, \omega_t\rangle$$

is conserved (cf. the proof of (5.3)). But by the unit propagation speed property (5.5), ω_t is supported in the complement of a ρ-neighbourhood of Crit(h) for $|t| < \rho$, and therefore $D_s^2 \omega_t = A\omega_t$. Thus ω_t for $|t| < \rho$ is also the unique solution to the equation

$$\frac{\partial^2 \omega_t}{\partial t^2} + A\omega_t = 0$$

with the same initial conditions, so we may write $\omega_t = \cos(t\sqrt{A})\omega$.

Now $\hat{\phi}$ has support in $[-\rho, \rho]$, and moreover is an even function (since ϕ is). Therefore

$$\begin{aligned}
\phi(D_s)\omega &= \frac{1}{2\pi}\int_{-\rho}^{\rho} (e^{itD_s}\omega)\hat{\phi}(t)\,dt \\
&= \frac{1}{\pi}\int_0^{\rho} \hat{\phi}(t)\cos(tD_s)\omega\,dt \\
&= \frac{1}{\pi}\int_0^{\rho} \hat{\phi}(t)\omega_t\,dt \\
&= \frac{1}{\pi}\int_0^{\rho} \hat{\phi}(t)\cos(t\sqrt{A})\omega\,dt \\
&= \dots = \phi(\sqrt{A})\omega.
\end{aligned}$$

This proves our claim. But now notice that $\sqrt{A}$ is a positive operator, bounded below by $\frac{1}{2}Cs$. It follows from the spectral theorem, then, that the L^2 operator norm of $\phi(\sqrt{A})$ is bounded above by

$$c(s) = \sup\{|\phi(\lambda)| : \lambda \geq \tfrac{1}{2}Cs\}.$$

As $s \to \infty$, this quantity tends to zero with rapid decay. So we deduce that if ω is supported in the complement of a 2ρ-neighbourhood of Crit(h),

$$\|\phi(D_s)\omega\| \leq c(s)\|\omega\| \qquad (\dagger)$$

with $c(s) \to 0$ rapidly as $s \to \infty$.

This is nearly what we want. In fact, if we can show that there is a $c_1(s)$, tending to zero as $s \to \infty$, with

$$\|\phi(D_s)\omega\|_{L^\infty} \leq c_1(s)\|\omega\|_{L^1} \qquad (\dagger\dagger)$$

(under the same condition on supp(ω)), we will be done, since for any integral operator with continuous kernel the supremum of the kernel can be estimated by the norm of the operator as a map from L^1 to L^∞; this is simply a rephrasing of the fact that $(L^1)^* = L^\infty$.

To get the improved estimate ($\dagger\dagger$) from ($\dagger$), we rely on the familiar techniques of Sobolev embedding. The key point is this: for any k, the operator $(1+D_s^2)^{-1}$ is bounded as an operator from W^k to W^{k+2}, with norm bounded by a polynomial in s. This follows from the elliptic estimates for D_s. Now by Sobolev embedding (3.7), $W^p \subset L^\infty$ for $p > \frac{n}{2}$, and therefore $(1+D_s^2)^{-k}$ is bounded from L^2 to L^∞ for $k > \frac{n}{4}$, the bound being polynomial is s. By duality and selfadjointness, $(1+D_s^2)^{-k}$ is also bounded (polynomially in s) as an operator from L^1 to L^2.

We deduce that the norm of $\phi(D_s)$ acting as an operator from L^1 to L^∞ is bounded by a polynomial in s times the norm of $(1+D_s^2)^{2k}\phi(D_s)$ acting as an operator on L^2. But this operator is just $\tilde{\phi}(D_s)$, where $\tilde{\phi}(\lambda) = (1+\lambda^2)^{2k}\phi(\lambda)$; the function $\tilde{\phi}$ satisfies the same conditions as ϕ, so

$$\|\tilde{\phi}(D_s)\omega\| \leq \tilde{c}(s)\|\omega\|$$

provided that ω satisfies the support condition, with $\tilde{c}(s)$ of rapid decay in s. Therefore

$$\|\phi(D_s)\omega\|_{L^\infty} \leq c_1(s)\|\omega\|_{L^1}$$

with $c_1(s) = \tilde{c}(s) \times$ (polynomial in s), which tends to zero as $s \to \infty$. □

From Lemma (12.10) it follows that as $s \to \infty$, the trace of $\phi(D_s)$ is given by a sum of contributions from the critical points of h. The reader should notice the similarity with the Lefschetz theorem (Chapter 8). We will now evaluate the contributions from the critical points.

In order to do this it is convenient to make a special choice of metric on our manifold m. This choice of metric uses the *Morse lemma.*

(12.11) Lemma: There are local co-ordinates (x^j) centered at each critical point of h with the property that in terms of these local co-ordinates h is a diagonal quadratic form,

$$h(x) = \tfrac{1}{2}\Sigma\lambda_j(x^j)^2.$$

Of course, the number of negative λ's is just the index of the critical point.

We will not go through the proof of the Morse lemma here. Proofs may be found in Milnor [46] or Guillemin and Sternberg [38].

(12.12) Now we choose our special metric g on M as follows; g is defined to be flat Euclidean ($g_{ij} = \delta_{ij}$) in Morse co-ordinates near each critical point, and is patched up away from Crit(h) using a partition of unity. We choose ρ so small that g is flat Euclidean at least to distance 4ρ from each critical point.

(12.13) Lemma: In a flat Euclidean metric in Morse co-ordinates

$$D_s^2 = \sum_j \left\{-\left(\frac{\partial}{\partial x^j}\right)^2 + s^2\lambda_j^2(x^j)^2 + s\lambda_j Z_j\right\}$$

where

$$Z_j = [dx^j \lrcorner, dx^j \wedge].$$

Proof: Clearly $D^2 = -\sum_j \left(\frac{\partial}{\partial x^j}\right)^2$, $H^2 = \sum \lambda_j^2 (x^i)^2$ by Lemma (12.8). We must check that $DH + HD = \Sigma\lambda_j Z_j$. Now

$$DH + HD = \sum_{i,j} (dx^j \wedge + dx^j \lrcorner) \frac{\partial}{\partial x^j} (\lambda_i x^i)(dx^i \wedge - dx^i \lrcorner)$$

$$= \sum_j \lambda_j (dx^j \wedge + dx^j \lrcorner)(dx^j \wedge - dx^j \lrcorner)$$

$$= \sum_j \lambda_j [dx^j \lrcorner, dx^j \wedge]. \qquad \square$$

Now let L_s denote the operator $\sum_j \left\{ -\left(\frac{\partial}{\partial x^j}\right)^2 + s^2\lambda_j^2 (x^j)^2 + s\lambda_j Z_j \right\}$ on differential forms over Euclidean space $\mathbb{R}^n$, so that L_s is our model for D_s^2 in a neighbourhood of a critical point.

(12.14) Lemma: L_s is an essentially self-adjoint operator, with discrete spectrum. The eigenvalues of L_s are the numbers

$$s \sum_j (|\lambda_j|(1 + 2p_j) + \lambda_j q_j)$$

where $p_j = 0,1,2,\ldots$, $q_j = \pm 1$. If we consider the action of L_s on k-forms, the spectrum is as above with the additional restriction that exactly k of the q_j's are equal to +1.

Proof: Notice that

$$Z_j dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \begin{cases} -dx^{i_1} \wedge \ldots \wedge dx^{i_k} & \text{if } j \notin \{i_1,\ldots,i_k\} \\ +dx^{i_1} \wedge \ldots \wedge dx^{i_k} & \text{if } j \in \{i_1,\ldots,i_k\}. \end{cases}$$

Let us write $Y_j = -\left(\frac{\partial}{\partial x^j}\right)^2 + s^2\lambda_j^2(x^j)^2$, so that Y_j is a harmonic oscillator (10.30) in the j-variable. The Z and Y operators all commute, so they can be "simultaneously diagonalised". By the spectral theory of the harmonic oscillator (10.34) - (10.36), we know that ΣY_j is essentially self-adjoint, with discrete spectrum: its eigenvalues are the numbers $s\Sigma|\lambda_j|(1 + 2p_j)$, and each of these eigenvalues has multiplicity 2^n (the fibre dimension of the exterior bundle). The operators Z_j act on each of the eigenspaces as involutions, splitting them into ±1 eigenspaces for each Z_j: the eigenspace with eigenvalue $s\Sigma(|\lambda_j|(1 + 2p_j) + \lambda_j q_j)$ for L is precisely the q_j-eigenspace for each Z_j acting on the $s\Sigma|\lambda_j|(1 + 2p_j)$-eigenspace for ΣY_j. □

(12.15) Lemma: Suppose that precisely m of the λ_j's are negative. Then

$$\lim_{s\to\infty} \mathrm{Tr}(\phi(\sqrt{L_s})_{|\Lambda^k}) = \begin{cases} 0 & (k \neq m) \\ 1 & (k = m) . \end{cases}$$

Moreover, the same limit holds good for $\mathrm{Tr}(B\phi(\sqrt{L_s})_{|\Lambda^k})$ where B is the operator of multiplication on $\mathbb{R}^n$ by any $\beta \in C_c^\infty(R^n)$ with $\beta(0) = 1$.

Proof: By (6.7) and (12.14)

$$\mathrm{Tr}(\phi(\sqrt{L_s})_{|\Lambda^k}) = \sum_{p_j,q_j} \phi\left(\sqrt{s\sum_j (|\lambda_j|(1 + 2p_j) + \lambda_j q_j)}\right)$$

where the summation is over $p_j = 0,1,2,\ldots$, $q_j = \pm 1$, and exactly k of the q_j's equal +1. If $k \neq m$, then all the eigenvalues of L_s are of order s. Since ϕ is rapidly decreasing, it is easy to check that the sum tends to 0 as $s \to \infty$. On the other hand, if k = m then precisely one eigenvalue equals 0 and the others are of order s. The 0-eigenvalue contributes 1 to the sum and the sum of the remaining terms tends to 0, for the same reason as before.

In the case of the more general trace $\mathrm{Tr}(B\phi(\sqrt{L_s})_{|\Lambda^k})$, let $e(p_j,q_j)$ denote the normalised eigenform of L_s corresponding to (p_j,q_j). Then (6.6) gives

$$\mathrm{Tr}(B\phi(\sqrt{L_s})_{|\Lambda^k}) = \sum_{p_j,q_j} \phi\left(\sqrt{s\sum_j(|\lambda_j|(1+2p_j)+\lambda_j q_j)}\right)\langle Be(p_j,q_j),e(p_j,q_j)\rangle.$$

For the same reason as before, only the zero eigenvalue makes a contribution to this sum that does not vanish as $t \to \infty$. The corresponding eigenform e_0 is just the ground state eigenfunction of the harmonic oscillator multiplied by a certain constant differential form; namely, by $dx^1 \wedge \ldots \wedge dx^m$ if we assume that the first m of the λ_j's are negative and the rest are positive. That is, by (10.32), the eigenform e_0 is given explicitly by

$$e_0 = (s^{n/2}\pi^{-n/4} \prod_j \lambda_j^{\frac{1}{2}})\exp(-s \sum_j \lambda_j (x^j)^2/2)dx^1 \wedge \ldots \wedge dx^m.$$

It is now easy to check explicitly that as $s \to \infty$, $\langle Be_0, e_0\rangle \to 1$; so the stated result follows. □

We can now state and prove Morse's theorem.

(12.16) Theorem: Let h be a Morse function on the compact manifold M. Let β_j denote the j'th Betti number of M, and let ν_j denote the number of critical points of h of index j. Then

$$\beta_0 \leq \nu_0$$

$$\beta_1 - \beta_0 \leq \nu_1 - \nu_0$$

$$\beta_2 - \beta_1 + \beta_0 \leq \nu_2 - \nu_1 + \nu_0$$

$$\cdot \quad \cdot \quad \cdot$$

$$\Sigma(-1)^j \beta_j = \Sigma(-1)^j \nu_j.$$

Proof: Choose a metric on M and a function ϕ in accordance with (12.12), (12.8). By (12.3), the Betti numbers β_j^s of Witten's perturbed de Rham complex satisfy the Morse inequalities with respect to the numbers $\mu_j^s = \mathrm{Tr}(\phi(D_s)_{|\Lambda^j})$.

But the perturbed de Rham complex is conjugate to the unperturbed one; the two complexes therefore have isomorphic homology groups. In particular $\beta_j^s = \beta_j$ for all j. The proof will therefore be completed if we can show that $\mu_j^s \to \nu_j$ as $s \to \infty$.

By (6.12), the trace of $\phi(D_s)_{|\Lambda^j}$ is obtained by integration of the local trace of the smoothing kernel of this operator over the diagonal. By (12.10), this local trace tends uniformly to zero except on a 2ρ-neighbourhood of each critical point. So the limit as $s \to \infty$ of $\phi(D_s)_{|\Lambda^j}$ is a sum of contributions from the critical points. The contribution from a critical point can be written $\lim_{s\to\infty} \mathrm{Tr}(B\phi(D_s)_{|\Lambda^j})$, where B is the multiplication operator by a smooth function on M equal to 1 on a 2ρ-neighbourhood of the critical point and supported in a 3ρ-neighbourhood of the critical point.

Now take Morse co-ordinates around the critical point. These enable us to identify forms supported on a 4ρ-neighbourhood of the critical point in M with forms supported on a 4ρ-neighbourhood of the origin in $\mathbb{R}^n$. Under this identification, D_s^2 corresponds to L_s.

Now a unit propagation speed argument exactly analogous to that given in (12.10) shows that

$$\phi(D_s)\alpha = \phi(\sqrt{L_s})\alpha$$

provided that α is supported in a 3ρ-neighbourhood of the critical point. Hence

$$\mathrm{Tr}(B\phi(D_s)_{|\Lambda^j}) = \mathrm{Tr}(B\phi(\sqrt{L_s})_{|\Lambda^j}).$$

But by Lemma (12.15), as $s \to \infty$, $\mathrm{Tr}(B\phi(\sqrt{L_s})_{|\Lambda^j})$ tends to 1 if the critical point has index j, and otherwise to 0. The result follows. □

Notes for Chapter 12

For the classical approach to Morse theory, see Milnor [46]. We have followed Witten [69] more or less, while avoiding the perturbation expansion. The finite-dimensional Morse theory described in this chapter should be thought of as a preliminary stage in the application of Morse theory to infinite-dimensional manifolds such as loop-spaces.